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AN ANALOGUE OF THE RAUCH VARIATIONAL FORMULA FOR PRYM DIFFERENTIALS

BY

ERIC ROBERT JABLOW

Department of Mathematics, State University of New York at Stony Brook, Stony Brook, NY 11794, USA

ABSTRACT

In [5], H. E. Rauch discovered a formula for the first variation of an abelian differential on a Riemann surface and its periods with respect to the change of complex structure induced by a Beltrami differential. R. S. Hamilton, in [3], and discussed by C. Earle in [1], found an elegant proof of the formula using only first principles and not requiring uniformization theory. His proof uses a small amount of Hodge theory, the Riemann bilinear period relations, and a simple operator construction. In this article, we find an analogue of Rauch's formula for the Prym differentials using some of Hamilton's techniques, the Hodge theorem for vector bundles, and the "Prym version" of the Riemann bilinear relations. We discover a complicated set of formulas for the variation of the Prym differentials, with different specific solutions depending to the make-up of the Prym character. We conclude that the variation of the Prym periods with a given character depends on the differentials for the character and the differentials for its inverse. This explains the simplicity of the classical case, where the character is its inverse.

1. Preliminary results on Prym differentials

Let X be a compact Riemann surface of genus $g \geqq 2$, let $\tilde{X}$ be its universal covering space, $p \in \tilde{X}$ be a distinguished point, and $A_1, \ldots, A_g, B_1, \ldots, B_g$ be a marking of X. We make the customary definition of C_j as the commutator $A_j B_j A_j^{-1} B_j^{-1}$. A *Prym representation* on X is a homomorphism $\rho : \pi_1(X, p) \to \mathbf{C}^*$. It is *normalized* if it is a character, if the image of ρ is contained in S^1. It is trivial if the image of every element is 1, if $\rho \equiv 1$.

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To each representation ρ corresponds a line bundle L_ρ over X. L_ρ is the line bundle whose factor of automorphy for each T is the number $\rho(T)$. If this bundle has a holomorphic section, then it is equivalent to the trivial bundle $X \times \mathbf{C}$. We call such a ρ *analytically trivial*. The analytically trivial representations are determined by Abel's theorem; however, any normalized analytically trivial representation is trivial [2, p. 124].

A *Prym differential* for the representation ρ is a 1-form τ on $\tilde{X}$ satisfying:

$$\tau(Tz) = \rho(T)\tau(z), \qquad \text{for all } T \in \pi_1(X, p). \tag{1}$$

If τ is a closed 1-form, then its *period* on the curve T is the integral $\tau(T) = \int_p^{T_p} \tau$. This integral depends on the base point p in the manner described below. These periods satisfy the following equations:

$$\tau(I) = 0, \tag{2a}$$

$$\tau(ST) = \tau(S) + \rho(S)\tau(T), \tag{2b}$$

$$\tau(T^{-1}) = -\rho(T)^{-1}\tau(T), \tag{2c}$$

$$\tau(STS^{-1}) = [1 - \rho(T)]\tau(S) + \rho(S)\tau(T), \tag{2d}$$

$$\tau(STS^{-1}T^{-1}) = [1 - \rho(T)]\tau(S) - [1 - \rho(S)]\tau(T), \tag{2e}$$

$$\tau(T) = 0 \qquad \text{if } T \in \pi_1(X, p)'', \tag{2f}$$

$$\int_q^{T_q} \tau = \tau(T) + [\rho(T) - 1] \int_p^q \tau. \tag{2g}$$

These results reveal the collection of periods of τ to an element of $Z^1(\pi_1(X, p), \rho)$, the 1-cocycles of the fundamental group with coefficients ρ. It is easy to see that $\int_p^q \tau$ can take any desired value; the functions of the form $T \mapsto c[\rho(T) - 1]$ form the 1-coboundaries $B^1(\pi_1(X, p), \rho)$. To avoid any confusion by our choice of base point, we can associate to τ any indefinite integral on $\tilde{X}$ we wish, as long as we do so consistently under addition of Prym differentials; this corresponds to associating to any τ a particular choice of representative for its class in $H^1(\pi_1(X, p), \rho)$. Notice that if $\rho(T) = 1$, $\tau(T)$ is unaffected by any change of base point. We will make a final choice later.

We now define the following sets: $\Lambda^1(X, \rho)$ is the set of C^∞ Prym differentials for ρ on X, $\Lambda^{1,0}(X, \rho)$ is the set of C^∞ Prym differentials of type $(1, 0)$ for ρ on X, $\Lambda^{0,1}(X, \rho)$ is the set of C^∞ Prym differentials of type $(0, 1)$ for ρ on X, $H^{1,0}(X, \rho)$ is the set of holomorphic Prym differentials for ρ on X, $H^{0,1}(X, \rho)$ is the set of antiholomorphic Prym differentials for ρ on X, and $K^1(X,\rho)$ is the set of harmonic Prym differentials for ρ on X. As each of these sets can be defined locally, they are all well-defined. Note that if $\tau \in \Lambda^1(X,\rho)$, then $\bar{\tau} \in \Lambda^1(X,\rho^{-1})$, by conjugation of equation (1) and the equality $\bar{\rho} = \rho^{-1}$. We also define $C^\infty(X,\rho)$ as the set of C^∞ functions f on $\tilde{X}$ satisfying $f(Tz) = \rho(T)f(z)$ for all T.

We can now discuss the equations for the Prym differentials corresponding to the Riemann bilinear relations for ordinary differentials. Let χ be a fundamental polygon for X. Then, $\partial\chi \sim I$ and $\partial\chi = C_1 \cdots C_g$. We then have the following:

LEMMA 1. $\Sigma_{j=1}^{g}\{[1-\rho(B_j)]\tau(A_j) - [1-\rho(A_j)]\tau(B_j)\} = 0.$

PROOF. Since each C_j is homologous to zero, $\rho(C_j) = 1$. Then,

$$\begin{aligned} 0 &= \tau(I) \\ &= \tau(C_1 \cdots C_g) \\ &= \tau(C_1) + \rho(C_1)\tau(C_2 \cdots C_g) \\ &= \tau(C_1) + \cdots + \tau(C_g) \\ &= [1-\rho(B_1)]\tau(A_1) - [1-\rho(A_1)]\tau(B_1) + \cdots + [1-\rho(B_g)]\tau(A_g) \\ &\quad - [1-\rho(A_g)]\tau(B_g). \end{aligned}$$

■

This is the universal constraint on the periods of closed Prym differentials; the holomorphic differentials have further constraints. We now state without full proof the Prym bilinear relations for closed Prym differentials, the equivalent of the Riemann bilinear relations for these Prym differentials. The problem is that the product of two elements of $\Lambda^1(X,\rho)$ is not generally a 2-form on X; we therefore consider the exterior product of an element of $\Lambda^1(X,\rho)$ with an element of $\Lambda^1(X,\rho^{-1})$. We may plausibly take exterior products of two elements of $\Lambda^1(X,\rho)$ only if $\rho^2 \equiv 1$, which corresponds to the classical case of Prym differentials. Fortunately, as we have mentioned before, the conjugate of an element of $\Lambda^1(X,\rho)$ is an element of $\Lambda^1(X,\rho^{-1})$.

PROPOSITION 1. *If* $\tau\in\Lambda^1(X,\rho)$ *and* $\sigma\in\Lambda^1(X,\rho^{-1})$ *are closed, then*

$$\begin{aligned}\iint_X \tau\wedge\sigma &= \sum_{j=1}^{g}\,[\rho(B_j)-1+\rho(A_j)^{-1}]\tau(A_j)\sigma(B_j)\\ &\quad-\sum_{j=1}^{g}\rho(B_j)^{-1}\rho(A_j)\tau(B_j)\sigma(A_j)+\sum_{j=1}^{g}\,[1-\rho(B_j)^{-1}]\tau(A_j)\sigma(A_j)\\ &\quad-\sum_{j=1}^{g}\,[1-\rho(A_j)]\tau(B_j)\sigma(B_j)+\sum_{1\leqq j<k\leqq g}\tau(C_j)\sigma(C_k).\end{aligned}\tag{3}$$

For the remainder of this paper, we restrict ourselves to nontrivial Prym characters ρ with $\rho(A_j)=1$ for all j. In this case, $\tau(C_j)=[1-\rho(B_j)]\tau(A_j)$, and the above results reduce to

$$\sum_{j=1}^{g}\,[1-\rho(B_j)]\tau(A_j)=0,\tag{4}$$

and

$$\begin{aligned}\iint_X \tau\wedge\sigma &= \sum_{j=1}^{g}\rho(B_j)\tau(A_j)\sigma(B_j)-\sum_{j=1}^{g}\rho(B_j)^{-1}\tau(B_j)\sigma(A_j)\\ &\quad+\sum_{j=1}^{g}\,[1-\rho(B_j)^{-1}]\tau(A_j)\sigma(A_j)\\ &\quad+\sum_{1\leqq j<k\leqq g}[1-\rho(B_j)][1-\rho(B_k)^{-1}]\tau(A_j)\sigma(A_k).\end{aligned}\tag{5}$$

The Riemann bilinear period inequality can now be adapted to the Prym differentials as follows:

PROPOSITION 2. *Let* τ *be an element of* $\mathrm{H}^{1,0}(X,\rho)$. *Then,* $i\iint_X\tau\wedge\bar\tau$ *is real and nonnegative, and*

$$\begin{aligned}0 &\leqq i\iint_X \tau\wedge\bar\tau\\ &= i\sum_{j=1}^{g}\rho(B_j)^{-1}\tau(A_j)\bar\tau(B_j - i\sum_{j=1}^{g}\rho(B_j)\tau(B_j)\bar\tau(A_j)\\ &\quad+i\sum_{j=1}^{g}\,[1-\rho(B_j)^{-1}]\,|\,\tau(A_j)|^2\\ &\quad+i\sum_{1\leqq j<k\leqq g}[1-\rho(B_j)][1-\rho(B_k)^{-1}]\tau(A_j)\bar\tau(A_k)\end{aligned}$$

$$= 2 \sum_{j=1}^{g} \Im[\rho(B_j)\bar{\tau}(A_j)\tau(B_j)] + i \sum_{j=1}^{g} [1 - \rho(B_j)^{-1}] |\tau(A_j)|^2$$

$$+ i \sum_{1 \leqq j < k \leqq g} [1 - \rho(B_j)][1 - \rho(B_k)^{-1}]\tau(A_j)\bar{\mathcal{T}}(A_k)$$

with equality if and only if τ *is identically* 0.

PROOF. Locally, $\tau = h(z)dz$, and so

$$\tau \wedge \bar{\tau} = |h(z)|^2 dz \wedge d\bar{z} = -2i |h(z)|^2 dx \wedge dy.$$

Thus, $i \iint_X \tau \wedge \bar{\tau} \geqq 0$. This is zero if and only if τ is identically zero. The remainder comes from the Prym bilinear relations. ∎

PROPOSITION 3. *If* $\tau \in \mathrm{H}^{1,0}(X, \rho)$, *and if* $\tau(A_j) = 0$ *for all* j, $\tau \equiv 0$. *Thus, for any collection of numbers* $\{c_j\}$ *satisfying* $\Sigma_{j=1}^{g}[1 - \rho(B_j)]c_j = 0$, *there is a holomorphic* τ *satisfying* $\tau(A_j) = c_j$ *for all* j.

PROOF. If $\tau(A_j) = 0$ for all j, we may apply the above proposition. We learn that $\iint_X \tau \wedge \bar{\tau} = 0$. As ρ is not analytically trivial, it is well known that the dimension of the space of holomorphic Prym differentials is $g - 1$. For example, we may use the Riemann-Roch theorem on KL_ρ^{-1} to show that there can be no further conditions on the $\tau(A_j)$ besides the one above. (Here K is the canonical line bundle.) ∎

We now establish a standard normalization for the periods of a Prym differential. Let us define the vector $\vec{e}_\rho$ as $(1 - \rho(B_1), \ldots, 1 - \rho(B_g))$. By equation 4, $\vec{e}_\rho$ is orthogonal to each vector $(\tau(A_1), \ldots, \tau(A_g))$. If this vector is not lightlike, if $\vec{e}_\rho \cdot \vec{e}_\rho \neq 0$, we can change our base point for τ so as to make $\vec{e}_\rho$ orthogonal to $(\tau(B_1), \ldots, \tau(B_g))$ also. This cannot affect $\tau(A_j)$ for any j. Actually, we choose the appropriate 1-cocycle for each Prym differential. With this description, all the problems with base points go away.

If $\vec{e}_\rho$ is lightlike, then we must change our marking to obtain a new homology basis $\langle[\tilde{A}_l], [\tilde{B}_l]\rangle$ with the vector $(1 - \rho(\tilde{B}_1), \ldots, 1 - \rho(\tilde{B}_g))$ not lightlike. For example, we can choose $j \leqq g - 1$ such that $\rho(B_j) \neq 1$. This is always possible; if no such j exists, then either this vector is not lightlike or ρ is trivial. We now choose an element of the modular group that acts on the canonical *homology* basis as follows:

$$[B_j] \mapsto [B_j] - [B_{j+1}],$$

$$[A_{j+1}] \mapsto [A_j] + [A_{j+1}],$$

$$[T] \mapsto [T] \qquad \text{if } [T] \neq [B_j], [A_{j+1}].$$

The map acting on the generators by

$$A_j \mapsto \tilde{A}_j = B_{j+1} A_j B_{j+1}^{-1},$$

$$B_j \mapsto \tilde{B}_j = B_{j+1} A_j^{-1} B_{j+1}^{-1} A_j B_j B_{j+1}^{-1},$$

$$A_{j+1} \mapsto \tilde{A}_{j+1} = B_{j+1} A_j^{-1} B_{j+1}^{-1} A_j A_{j+1} B_{j+1} A_j B_{j+1}^{-1},$$

$$B_{j+1} \mapsto \tilde{B}_{j+1} = B_{j+1} A_j^{-1} B_{j+1} A_j B_{j+1}^{-1},$$

$$T \mapsto \tilde{T} = T \qquad \text{if } T \neq A_j, B_j, A_{j+1}, \text{ or } B_{j+1},$$

has the desired action on the homology basis. After we change the marking by this map, we have $\rho(\tilde{A}_l) = 1$ for all l. We also have $\rho(\tilde{B}_j) = \rho(B_j)/\rho(B_{j+1})$ and $\rho(\tilde{B}_l) = \rho(B_l)$ for all $l \neq j$. Then, the values of a Prym differential τ on this modified marking is:

$$\tau(\tilde{A}_j) = \rho(B_{j+1})\tau(A_j),$$

$$\tau(\tilde{B}_j) = (1 - \rho(B_{j+1}))\tau(A_j) + \tau(B_j) - \rho(B_j)\rho(B_{j+1})^{-1}\tau(B_{j+1}),$$

$$\tau(\tilde{A}_{j+1}) = \rho(A_j) + \rho(A_{j+1}),$$

$$\tau(\tilde{B}_{j+1}) = \tau(B_{j+1}) + \rho(B_{j+1})(\rho(B_{j+1}) - 1)\tau(A_j),$$

$$\tau(\tilde{T}) = \tau(T) \qquad \text{if } T \neq A_j, B_j, A_{j+1}, \text{ or } B_{j+1}.$$

In fact, let us choose a basis for the holomorphic Prym differentials $\tau_1, \dots, \tau_{g-1}$. Any basis will be satisfactory, but we can choose it so that for $j \neq k$, $(\tau_j(A_1), \dots, \tau_j(A_g))$ is orthonormal to $(\tau_k(A_1), \dots, \tau_k(A_g))$. The reason for this will be apparent at the end. We define the *Prym matrix* as:

$$\begin{pmatrix} \tau_1(A_1) & \cdots & \tau_1(A_g) & \tau_1(B_1) & \cdots & \tau_1(B_g) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \tau_{g-1}(A_1) & \cdots & \tau_{g-1}(A_g) & \tau_{g-1}(B_1) & \cdots & \tau_{g-1}(B_g) \end{pmatrix}. \tag{6}$$

2. Beltrami differentials and Prym differentials

We recall the definition of a Beltrami differential on a Riemann surface X, and how a Beltrami differential μ induces a deformation of the Riemann surface into the Riemann surface X^μ. A C^∞ Beltrami differential on X, μ, is an object that can be expressed in terms of a local coordinate on X as follows:

(7) $$\mu = \mu(z)d\bar{z}/dz$$

where μ must be invariant under change of coordinates. In other words, if $\mu(z)$ is defined on $\tilde{X}$, then

(8) $$\mu(Tz) = \mu(z)\overline{T'(z)}/T'(z).$$

The absolute value of $\mu(z)$ is a well-defined real-valued function on X. We assume that the supremum $\| \mu \|$ is less than one. In this case, the Beltrami differential equation

(9) $$w_{\bar{z}}(z) = \mu(z)w_z(z)$$

has a C^∞ solution on X for any local coordinate z, and all solutions w are holomorphically related. The solutions w define an atlas on the underlying topological surface of X; since they are holomorphically related, they define a Riemann surface X^μ topologically equivalent to X. It is well known that every Riemann surface topologically equivalent to X is conformally equivalent to X^μ for some Beltrami differential μ. Furthermore, for Riemann surfaces near X in moduli space, we can assume μ to be C^∞.

Since μ is C^∞, X and X^μ have equivalent C^∞-structures. Thus any C^∞ 1-form on X^μ is a C^∞ 1-form on X, and any closed C^∞ 1-form on X^μ is closed on X. In particular, any abelian differential on X^μ is a closed 1-form on X. We can define sets $\Lambda^1(X)$, $\Lambda^{1,0}(X)$, $\Lambda^{0,1}(X)$, $H^{1,0}(X)$, and $H^{0,1}(X)$, with the obvious meanings. Now, any C^∞ 1-form ω on X has a unique expansion as a sum $\omega' + \omega''$, where $\omega' \in \Lambda^{1,0}(X)$, and $\omega'' \in \Lambda^{0,1}(X)$. We will prove this result for Prym differentials later. Furthermore, the product of a Beltrami differential with an element of $\Lambda^{1,0}(X)$ is an element of $\Lambda^{0,1}(X)$.

Suppose that ω is an abelian differential on X^μ. In terms of a local coordinate w on X^μ, $\omega = f(w)dw$ with f holomorphic. Since f must be a solution of the Beltrami equation,

$$dw = w_z dz + w_{\bar{z}} d\bar{z} = w_z(dz + \mu d\bar{z}),$$

and so $\omega = f(w(z))w_z(dz + \mu d\bar{z})$. Here, z is a local coordinate on X. Thus on X, $\omega' = f(w(z))w_z dz$, and $\omega'' = f(w(z))w_z\mu(z)d\bar{z}$, and so $\omega'' = \mu\omega'$. In fact, any closed 1-form ω on X such that $\omega'' = \mu\omega'$ is an abelian differential on X^μ.

Now, we repeat this discussion for the Prym differentials on X and X^μ.

Let τ be a C^∞ Prym differential for ρ on X^μ. Locally, it can be written uniquely as a sum of C^∞ 1-forms τ' and τ'', where τ' is of type $(1,0)$ and τ'' is of type $(0,1)$. These two differentials can be thought of as multi-valued 1-forms on X.

LEMMA 2. *Any C^∞ Prym differential τ on X^μ is uniquely the sum of a C^∞ Prym differential of type* $(1,0)$ *on X and a C^∞ Prym differential of type* $(0,1)$ *on X.*

PROOF. The local decomposition of τ is unique. Therefore,

$$\tau(z) = \tau'(z) + \tau''(z),$$

which implies

$$\tau(Tz) = \tau'(Tz) + \tau''(Tz).$$

Then

$$\rho(T)\tau(z) = \tau'(Tz) + \tau''(Tz),$$

and therefore

$$\tau(z) = \rho(T)^{-1}\tau'(Tz) + \rho(T)^{-1}\tau''(Tz).$$

Since the decomposition is unique,

$$\tau'(z) = \rho(T)^{-1}\tau'(T) \quad \text{and} \quad \tau''(z) = \rho(T)^{-1}\tau''(Tz).$$

Thus $\tau'(Tz) = \rho(T)\tau'(z)$ and $\tau''(Tz) = \rho(T)\tau''(z)$, and both τ' and τ'' are Prym. ■

Now assume τ is a holomorphic Prym differential on X^μ. Locally $\tau = h(w)dw$. Since w satisfies the Beltrami equation,

$$\tau = h(w(z))w_z dz + h(w(z))w_z \mu(z) d\bar{z},$$

and the two summands $h(w(z))w_z dz$ and $h(w(z))w_z\mu(z)d\bar{z}$ are Prym for ρ. thus, $\tau'' = \mu\tau'$, similar to the abelian case.

3. Dualities between Prym differentials

We now discuss an extension of a well-known result in the abelian case to the Prym case. It is a simple consequence of the Hodge theorem that any C^∞ differential form of type $(1,0)$ ϕ can be written uniquely as a sum $\psi + \partial f$, where ψ is a holomorphic differential and f is a C^∞ function on X. The

Hodge theorem does apply to differential forms with values in a vector field over a manifold [6, p. 147]; in our case, Prym differentials are differential forms with values in L_ρ, a holomorphic flat line bundle over X, and this allows us to show that exactly the same decomposition occurs for a C^∞ Prym differential τ.

The Hodge theorem for vector bundles [6, p. 168] as applied to this situation of a line bundle over a Riemann surface leads to the following:

$$\Lambda^1(X,\rho) = \mathrm{K}^1(X,\rho) \oplus \bar\partial C^\infty(X,\rho) \oplus \bar\partial^* \Lambda^2(X,\rho), \tag{10a}$$

where $\bar\partial^*$ is the adjoint of $\bar\partial$. The adjoint $\bar\partial^* = -\bar{*}_{L_\rho^*}\bar\partial\,\bar{*}_{L_\rho}$, where $\bar{*}_{L_\rho}$ is the Hodge star operator on L_ρ. It is defined as the map from $\Lambda^m T^*(X)\otimes L_\rho$ to $\Lambda^{2-m}\mathrm{T}^*(X)\otimes L_\rho^*$ given by $\bar{*}_{L_\rho}(\phi\otimes e) = *\phi\otimes e^*$, where $*\phi$ is the ordinary Hodge star, and e^* is the dual in the hermitian metrix of the fiber. Since the fiber of L_ρ is $\mathbf{C}$, we can guess that the hermitian metric is simply $\zeta\otimes\bar\zeta$; since the factor of automorphy for the change from z to Tz is $\rho(T)$, which takes $\zeta\otimes\bar\zeta$ to $\rho(T)\zeta\otimes\bar\rho(T)\bar\zeta$, which equals $\zeta\otimes\bar\zeta$, our guess is correct. Thus $e^* = \bar e$, and so $\bar{*}_{L_\rho^*}\bar\partial = \partial$, and we have the decomposition

$$\Lambda^1(X,\rho) = \mathrm{K}^1(X,\rho) \oplus \bar\partial C^\infty(X,\rho) \oplus \partial C^\infty(X,\rho). \tag{10b}$$

Since τ is a differential of type $(1,0)$, it therefore splits into the sum of a harmonic Prym differential of type $(1,0)$, hence holomorphic, and ∂h, for some $h\in C^\infty(X,\rho)$. Thus, $\tau = \alpha + \partial h$, the same as the abelian case. We use this to establish a duality between $\Lambda^{1,0}(X,\rho)$ and $\Lambda^{1,0}(X,\rho^{-1})$.

Proposition 4. *There exists a unique map* $T\colon \Lambda^{1,0}(X,\rho^{-1})\to\Lambda^{1,0}(X,\rho)$ *such that if* $\tau = T\sigma$ *then* $d(\tau+\bar\sigma) = 0$ *and* $\int_{A_j}(\tau+\bar\sigma) = 0$ *for all* j.

Proof. If both τ_1 and τ_2 satisfy these conditions, then

$$d(\tau_1-\tau_2) = d(\tau_1+\bar\sigma) - d(\tau_2+\bar\sigma) = 0.$$

Therefore, $(\tau_1-\tau_2)$ is holomorphic. Furthermore,

$$\int_{A_j}(\tau_1+\tau_2) = \int_{A_j}(\tau_1+\bar\sigma) - \int_{A_j}(\tau_2+\bar\sigma) = 0.$$

Therefore, by Proposition 3, $\tau_1-\tau_2 = 0$. Thus, any solution is unique.

To show the existence of a solution, we expand σ as a sum $\beta+\partial h$, where β is holomorphic, and $h\in\Lambda^1(X,\rho^{-1})$. Then, $\bar\sigma = \bar\beta + \bar\partial\bar h$, where $\bar\beta\in\mathrm{H}^{0,1}(X,\rho)$ and $\bar h\in C^\infty(X,\rho)$. Since the periods of σ satisfy

$$\sum_{j=1}^{g} [1-\rho(B_j)^{-1}]\beta(A_j)=0,$$

the periods of $\bar{\sigma}$ satisfy

$$\sum_{j=1}^{g} [1-\rho(B_j)]\bar{\beta}(A_j)=0.$$

By Proposition 3, there exists a holomorphic Prym differential α for ρ such that $\int_{A_j}\alpha=-\int_{A_j}\bar{\beta}$ for all j. Let $\tau=\alpha+\partial\bar{h}$. Then

$$\tau+\bar{\sigma}=\bar{\beta}+\alpha+\bar{\partial}\bar{h}+\partial\bar{h}=\bar{\alpha}+\beta+d\bar{h}.$$

This is the sum of an antiholomorphic Prym differential, a holomorphic Prym differential, and an exact Prym differential. Certainly it is closed. Obviously, its A-periods are obviously zero. Thus $\tau=T\sigma$.

Note that $T(\partial h)=\partial\bar{h}$. We let $L:\Lambda^{0,1}(X,\rho)\rightarrow\Lambda^{1,0}(X,\rho)$ be given by $L\gamma=T\bar{\gamma}$. Then $d(\gamma+L\gamma)=0$ and $\int_{A_j}\gamma+L\gamma=0$ for all j. Please also note that $T^2=L^2=\mathrm{id}$, meaning that the action of T on $\Lambda^{1,0}(X,\rho^{-1})$ followed by its action on $\Lambda^{1,0}(X,\rho)$ is the identity. ■

4. Comparison theorems for Prym differentials

In this section, we will prove that we can reconstruct the Prym differentials on X^μ from corresponding differentials on X and from the two operators T and L discussed above. In order to do this we will design an inner product structure on $\Lambda^{1,0}(X,\rho)$ and $\Lambda^{0,1}(X,\rho)$ for any Prym character ρ.

LEMMA 3. *The space $\Lambda^{1,0}(X,\rho)$, endowed with the product*

$$(\beta,\alpha)=i\iint_X\beta\wedge\bar{\alpha},$$

is an inner product space. The space $\Lambda^{0,1}(X,\rho)$, endowed with the product

$$(\beta,\alpha)=-i\iint_X\beta\wedge\bar{\alpha},$$

is an inner product space.

PROOF. It is obvious that these products are sesquilinear. Therefore, we need only show that

(1) $(\beta,\beta)\geqq 0$ with equality only when $\beta\equiv 0$.

(2) $(\alpha,\beta)=\overline{(\beta,\alpha)}$.

Part 1 is true since $(\beta, \beta) = i \iint_X \beta \wedge \bar{\beta}$, and the integrand is a well-defined C^∞ (1, 1)-form on X. If $\beta = h(z)dz$ locally (and here, h need not be holomorphic), then

$$
\begin{aligned}
i \iint_X \beta \wedge \bar{\beta} &= i \iint_X h(z)dz \wedge \bar{h}(z)d\bar{z} \\
&= i \iint_X |h(z)|^2 dz \wedge d\bar{z} \\
&= 2 \iint_X |h(z)|^2 dx \wedge dy \\
&\geqq 0.
\end{aligned}
$$

This equals zero if and only if h is identically zero. Equivalently, this is true if and only if β is identically zero. Part 2 is trivial, and the result for $\Lambda^{0,1}(X, \rho)$ is an easy consequence of this result. ■

Now, the only constraint on the A-periods of a holomorphic Prym differential is given by equation (4). This constraint depends only on the numbers $\rho(B_j)$, and not on the surface X. Therefore, given a Prym differential $\tau(0)$ on X, there exists a Prym differential $\tau(\mu)$ on X^μ with identical A-periods:

$$
\int_{A_j} \tau(\mu) = \int_{A_j} \tau(0) \qquad \text{for all } j. \tag{11}
$$

Now, write $\tau(\mu)$ as a sum $\tau(\mu)' + \tau(\mu)''$, where $\tau(\mu)' \in \Lambda^{1,0}(X, \rho)$ and $\tau(\mu)'' \in \Lambda^{0,1}(X, \rho)$. The differential $\tau(\mu) - \tau(0)$ is closed and has zero A-periods. Thus,

$$
d[\tau(\mu) - \tau(0)] = 0,
$$

$$
d\{[\tau(\mu)' - \tau(0)] + \tau(\mu)''\} = 0,
$$

and

$$
\int_{A_j} [\tau(\mu) - \tau(0)] = 0.
$$

Therefore,

$$
\int_{A_j} \{[\tau(\mu)' - \tau(0)] + \tau(\mu)''\} = 0 \qquad \text{for all } j.
$$

By the definition of L, $\tau(\mu)' - \tau(0) = \tau(\mu)''$. However, we have shown that $\tau(\mu)'' = \mu\tau(\mu)'$, and so:

$$\tau(\mu)' - \tau(0) = L(\mu\tau(\mu)'),$$

$$\tau(0) = \tau(\mu)' - L(\mu\tau(\mu)'), \tag{12}$$

$$\tau(0) = (I - L\mu)\tau(\mu)'.$$

If the operator $(I - L\mu)$ is invertible, then

$$\tau(\mu)' = (I - L\mu)^{-1}\tau(0),$$

and

$$\tau(\mu) = (I - L\mu)^{-1}\tau(0) + \mu(I - L\mu)^{-1}\tau(0).$$

THEOREM 1. *The mappings T and L are isometries. Therefore,* $\| L\mu \| = \| \mu \| < 1$, *and on the Hilbert space completion of* $\Lambda^{1,0}(X, \rho)$, $(I - L\mu)$ *has the inverse* $\Sigma_{n=0}^{\infty}(L\mu)^n$. *Consequently,* $\tau(\mu)' = \Sigma_{n=0}^{\infty}(L\mu)^n\tau(0)$, *and* $\tau(\mu)'' = \mu\,\Sigma_{n=0}^{\infty}(L\mu)^n\tau(0)$.

PROOF. Let $\sigma \in \Lambda^{1,0}(X, \rho^{-1})$, and let $\tau = T\sigma \in \Lambda^{1,0}(X, \rho)$. Then $\sigma = \beta + \partial h$, where β is a holomorphic Prym differential for ρ^{-1} and $h \in C^{\infty}(X, \rho^{-1})$. The differential $\tau = T\sigma$ has a representation as a sum $\alpha + \partial\bar{h}$, where $\alpha \in \mathrm{H}^{1,0}(X, \rho)$. We know that $\int_{A_j}\alpha = -\int_{A_j}\bar{\beta}$, or more simply, $\alpha(A_j) = -\bar{\beta}(A_j)$. We will show that $\| \beta \| = \| \alpha \|$, that $\| \partial h \| = \| \partial\bar{h} \|$, and that $(\beta, \partial h) = (\alpha, \partial\bar{h}) = 0$. This will imply that $\| \sigma \| = \| \tau \|$.

By the Prym bilinear relations,

$$\begin{aligned}
\| \beta \|^2 &= (\beta, \beta) \\
&= i \sum_{j=1}^{g} [\rho(B_j)^{-1}\beta(A_j)\bar{\beta}(B_j) - \rho(B_j)\beta(B_j)\bar{\beta}(A_j)] \\
&\quad + i \sum_{j=1}^{g} (1 - \rho(B_j))\beta(A_j)\bar{\beta}(A_j) \\
&\quad + i \sum_{1 \leq j < k \leq g} (1 - \rho(B_j)^{-1})(1 - \rho(B_k))\beta(A_j)\bar{\beta}(A_k)
\end{aligned}$$

and

$$\begin{aligned}\|\alpha\|^2 &= (\alpha, \alpha)\\ &= i\sum_{j=1}^{g}[\rho(B_j)\alpha A_j\bar{\alpha}(B_j) - \rho(B_j)^{-1}\alpha(B_j)\bar{\alpha}(A_j)]\\ &\quad + i\sum_{j=1}^{g}(1-\rho(B_j)^{-1})\alpha(A_j)\bar{\alpha}(A_j)\\ &\quad + i\sum_{1\leqq j<k\leqq g}(1-\rho(B_j))(1-\rho(B_k)^{-1})\alpha(A_j)\bar{\alpha}(A_k).\end{aligned}$$

We can now freely use the relations $\rho(B_j)^{-1} = \bar{\rho}(B_j)$ and $\alpha(A_j) = -\bar{\beta}(A_j)$ to get

$$\begin{aligned}\|\beta\|^2 - \|\alpha\|^2 = &-2\sum_{j=1}^{g}\Im[\rho(B_j)^{-1}\beta(A_j)\bar{\beta}(B_j)] + 2\sum_{j=1}^{g}\Im[\rho(B_j)\alpha(A_j)\bar{\alpha}(B_j)]\\ &-2\sum_{j=1}^{g}\Im(1-\rho(B_j))\|\beta(A_j)\|^2\\ &-2\sum_{1\leqq j<k\leqq g}\Im[(1-\rho(B_j)^{-1})(1-\rho(B_k))\beta(A_j)\bar{\beta}(A_k)].\end{aligned}$$

Meanwhile,

$$\begin{aligned}\Im\iint_X \beta\wedge\alpha = &\sum_{j=1}^{g}\Im[\rho(B_j)^{-1}\beta(A_j)\alpha(B_j)] - \sum_{j=1}^{g}\Im[\rho(B_j)\beta(B_j)\alpha(A_j)]\\ &+\sum_{j=1}^{g}\Im[(1-\rho(B_j))\beta(A_j)\alpha(A_j)]\\ &+\sum_{1\leqq j<k\leqq g}\Im[(1-\rho(B_j)^{-1})(1-\rho(B_k))\beta(A_j)\alpha(A_k)]\\ = &(\|\beta\|^2 - \|\alpha\|^2)/2,\end{aligned}$$

as $\Im\bar{z} = -\Im z$ and $\alpha(A_j) = -\bar{\beta}(A_j)$. However, β and α are forms of type (1, 0), and so $\iint_X \beta\wedge\alpha = 0$. Thus $\|\beta\|^2 = \|\alpha\|^2$.

To show $\|\partial h\| = \|\partial\bar{h}\|$, we consider that

$$\begin{aligned}i\iint_X dh\wedge d\bar{h} &= i\iint_X(\partial h + \bar{\partial}h)\wedge(\partial\bar{h} + \bar{\partial}\bar{h})\\ &= i\iint_X(\partial h\wedge\bar{\partial}\bar{h}) + i\iint_X(\bar{\partial}h\wedge\partial\bar{h})\\ &= i\iint_X \partial h\wedge\bar{\partial}\bar{h} - i\iint_X \partial\bar{h}\wedge\bar{\partial}h\end{aligned}$$

$$= \| \partial h \|^2 - \| \partial \bar{h} \|^2.$$

Since dh and $d\bar{h}$ are closed and $\rho(A_j) = 1$ for all j, $\int_{A_j} dh = \int_{A_j} d\bar{h} = 0$. Therefore $i \iint_X dh \wedge d\bar{h} = 0$, and $\| \partial h \| = \| \partial \bar{h} \|$ by the Prym bilinear relations. In fact, we may consider what the correct normalization of the periods of dh must be. Suppose the correct antiderivative of dh has value c at p. Then $\int_{A_j} dh = 0$, and $\int_{B_j} dh = c(\rho(B_j) - 1)$. In order for $\vec{e}_\rho$ to be orthogonal to $(\int_{B_1} dh, \ldots, \int_{B_g} dh)$, c must equal 0. Thus all the periods of dh are 0.

Finally, we show that β is orthogonal to ∂h:

$$\begin{aligned}(\beta, \partial h) &= i \iint_X \beta \wedge \bar{\partial} \bar{h} \\ &= i \iint_X \beta \wedge \bar{\partial} \bar{h} + i \iint_X \beta \wedge \partial \bar{h} \\ &= i \iint_X \beta \wedge d\bar{h} \\ &= -i \iint_X d\bar{h} \wedge \beta \\ &= 0, \qquad \text{as all the periods of } dh \text{ are } 0.\end{aligned}$$

By the same argument, $(\alpha, \partial \bar{h}) = 0$, and so

$$\begin{aligned}(\sigma, \sigma) &= (\beta + \partial h, \beta + \partial h) \\ &= (\beta, \beta) + (\beta, \partial h) + (\partial h, \beta) + (\partial h, \partial h) \\ &= (\beta, \beta) + 0 + 0 + (\partial h, \partial h) \\ &= (\alpha, \alpha) + (\alpha, \partial \bar{h}) + (\partial \bar{h}, \alpha) + (\partial \bar{h}, \partial \bar{h}) \\ &= (\tau, \tau) \\ &= (T\sigma, T\sigma).\end{aligned}$$

Therefore, T is an isometry. Since $L\gamma = T\bar{\gamma}$ and $\| \gamma \| = \| \bar{\gamma} \|$, L is an isometry also. ∎

We conclude that

$$\tau(\mu) = \tau(0) + \sum_{n=1}^{\infty} (L\mu)^n \tau(0) + \mu \sum_{n=0}^{\infty} (L\mu)^n \tau(0). \tag{13}$$

5. Variations of Prym periods

Let $\omega_1, \ldots, \omega_g$ be a basis for the abelian differentials of X dual to the given marking; then the Rauch variational formula for the period ω_{jk} is:

$$\omega_{jk}(\mu) = \omega_{jk}(0) + \iint_X \omega_j(0) \wedge \omega_k(\mu),$$

$$\begin{aligned}\omega_{jk}(\mu) &= \omega_{jk}(0) + \iint_X \omega_j(0) \wedge \omega_k(\mu)'' \\ &= \omega_{jk}(0) + \iint_X \omega_j(0) \wedge \mu \sum_{n=0}^{\infty} (L\mu)^n \omega_k(0) \\ &= \omega_{jk}(0) + \iint_X \omega_j(0) \wedge \mu\omega_k(0) + O(\|\mu\|^2).\end{aligned} \tag{14}$$

The corresponding formula for the Prym periods is much more difficult to obtain for two reasons. First, the left half of the abelian period matrix can be chosen to be I_g; there is no standard choice for the left half of the Prym period matrix. For example, if the numbers $\rho(B_1), \ldots, \rho(B_g)$ contain at least two distinct values other than 1, there must be at least one row with complex entries. Thus, any formula must consider a general choice of the left half of the Prym matrix. Second, one can consider an integral of the wedge product of two elements of $\Lambda^{1,0}(X, \rho)$ only if $\rho = \rho^{-1}$; we discuss this case in the next section. It includes the case of the classical Prym periods.

We are given a basis for the Prym differentials for ρ on X, say τ_j, for $1 \leqq j \leqq g-1$, where the rows of each half of the Prym matrix are orthogonal to $\vec{e}_\rho$. We take the liberty of writing τ_j instead of $\tau_j(0)$ above. Then, we find the right half of the Prym matrix of $\tau_1(\mu), \ldots, \tau_{g-1}(\mu)$ on X^μ as follows: remember that T acts on every space $\mathrm{H}^{1,0}(X, \rho)$, and $T^2 = \mathrm{id}$. Then as $T\tau_j$ and τ_k are holomorphic,

$$\begin{aligned}\iint_X T\tau_j \wedge \tau_k(\mu) &= \iint_X T\tau_j \wedge \tau_k(\mu) - \iint_X T\tau_j \wedge \tau_k \\ &= \iint_X T\tau_j \wedge (\tau_k(\mu) - \tau_k) \\ &= \sum_{l=1}^{g} \rho(B_l)^{-1} T\tau_j(A_l)[\tau_k(\mu)(B_l) - \tau_k(B_l)] \\ &= \sum_{l=1}^{g} -\rho(B_l)^{-1} \bar{\tau}_j(A_l)[\tau_k(\mu)(B_l) - \tau_k(B_l)].\end{aligned}$$

However,

$$\int\!\!\int_X T\tau_j \wedge \tau_k(\mu) = \int\!\!\int_X T\tau_j \wedge (\tau_k(\mu)' + \tau_k(\mu)'')$$
$$= \int\!\!\int_X T\tau_j \wedge (\tau_k(\mu)''$$
$$= \int\!\!\int_X T\tau_j \wedge \mu \sum_{n=0}^{\infty} (L\mu)^n \tau_k$$
$$= \int\!\!\int_X T\tau_j \wedge \mu\tau_k + O(\|\mu\|^2).$$

We can write these equations in matrix form:

THEOREM 2.

$$-(\rho(B_l)^{-1}\bar{\tau}_j(A_l))_{j,l} \cdot (\tau_k(\mu)(B_l) - \tau_k(B_l))_{l,k} = \left(\int\!\!\int_X T\tau_j \wedge \mu\tau_k\right)_{j,k} + O(\|\mu\|^2). \tag{15}$$

Now, the product of a $g-1$ by g matrix by a g by $g-1$ matrix equals a $g-1$ by $g-1$ matrix. Can we solve for the matrix $(\tau_k(\mu)(B_l) - \tau_k(B_l))_{l,k}$? Yes, as this matrix maps $\mathbf{C}^{g-1}$ into the hyperplane $\vec{e}_\rho \cdot (z_1, \ldots, z_g) = 0$, and the kernel of the matrix $(\rho(B_l)^{-1}\bar{\tau}_j(A_l))_{j,l}$ is just the set of multiples of $\vec{e}_\rho$. We cannot give a general formula for the matrix $(\tau_k(\mu)(B_l) - \tau_k(B_l))_{l,k}$, but we can always solve for it.

It is of special interest that in order for us to know the variation of $\tau_j(B_k)$, we must know all the differentials $\tau_1, \ldots, \tau_{g-1}$. This is caused by the impossibility of defining a canonical left half of the Prym period matrix. More interesting is that for us to know the variation of the differentials τ_j and of the periods $\tau_j(B_k)$, we must know the differentials $T\tau_j$. We must know the structure of *both* $\mathrm{H}^{1,0}(X,\rho)$ *and* $\mathrm{H}^{1,0}(X,\rho^{-1})$ to know these variations. When $\rho^2 \equiv 1$, these two spaces are identical, and we have some remarkable simplifications.

6. The classical Prym differentials

When $\rho^2 = 1$, $\rho = \bar{\rho}$, and so $\Lambda^1(X,\rho) = \Lambda^1(X,\rho^{-1})$. Thus T is an automorphism of $\Lambda^{1,0}(X,\rho)$ and of $\mathrm{H}^{1,0}(X,\rho)$. It is only real-linear; it satisfies $T(c\tau) = \bar{c}T(\tau)$. However, it can be calculated easily, as ρ can only take on the values 1 and -1, and so the components of the vector $\vec{e}_\rho$ are 0 and 2 only. In this case, there exists a basis $\tau_1, \ldots, \tau_{g-1}$ for the Prym differentials such that every entry in the left half of the period matrix is real. Then, $T\tau_j = -\tau_j$ for all j, and the matrix equations can be heavily simplified. We consider the two simplest cases here.

6.1. *The classical* 2–1 *Prym cover*

Let ρ be given by $\rho(A_j) = 1$, $\rho(B_1) = -1$, and $\rho(B_j) = 1$, for $2 \leqq j \leqq g$. This corresponds to the following: let $\hat{X}$ be the $2-1$ unbranched cover of X induced by the normal closure of B_1^2 in $\pi_1(X)$. Then, $\hat{X}$ has an involution induced by the action of B_1; this in turn induces a linear transformation on $H^{1,0}(\hat{X})$, and the projection of the -1-eigenspace of this map to X gives the Prym differentials. Since $\vec{e}_\rho = (2, 0, \ldots, 0)$, the Prym matrix can be written as:

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & \tau_1(B_2) & \cdots & \tau_1(B_g) \\ 0 & 0 & 1 & \cdots & 0 & 0 & \tau_2(B_2) & \cdots & \tau_2(B_g) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & \tau_{g-1}(B_2) & \cdots & \tau_{g-1}(B_g) \end{bmatrix}. \tag{16a}$$

The shortened matrix

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & \tau_1(B_2) & \cdots & \tau_1(B_g) \\ 0 & 1 & \cdots & 0 & \tau_2(B_2) & \cdots & \tau_2(B_g) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \tau_{g-1}(B_2) & \cdots & \tau_{g-1}(B_g) \end{bmatrix} \tag{16b}$$

is the matrix of the Prym variety; the right half is symmetric and has positive definite imaginary part, because:

$$\begin{aligned} 0 &= \iint_X \tau_j \wedge \tau_k \\ &= \sum_{l=1}^{g} \rho(B_l)^{-1}\tau_j(A_l)\tau_k(B_l) - \sum_{l=1}^{g} \rho(B_l)\tau_j(B_l)\tau_k(A_l) \\ &\quad + \sum_{l=1}^{g} [1-\rho(B_l)^{-1}]\tau_j(A_l)\tau_k(A_l) \\ &\quad + \sum_{1 \leqq l < m \leqq g} [1-\rho(B_l)][1-\rho(B_m)^{-1}]\tau_j(A_l)\tau_k(A_m) \\ &= \tau_k(B_{j+1}) - \tau_j(B_{k+1}). \end{aligned}$$

Similarly, $i \iint_X \tau_j \wedge \bar{\tau}_k = 2\Im \tau_j(B_{k+1})$, and so if we let $\tau = \Sigma\, c_j \tau_j$,

$$0 \leqq i \iint_X \tau \wedge \bar{\tau}$$

$$= 2 \sum_{j=1}^{g-1} \sum_{k=1}^{g-1} c_j \bar{c}_k \Im \tau_j(B_{k+1}),$$

and so the imaginary part of the right half of the matrix of the Prym variety is positive definite. The lattice generated by the columns of this matrix is the Prym lattice, and $\mathbf{C}^{g-1}$ divided by the Prym lattice is the Prym variety.

Now, we determine the variation of the Prym periods. With this basis for the Prym differentials, $T\tau_j = -\tau_j$, and so (where $\Delta\tau_j(B_k)$ is $\tau_j(\mu)(B_k) - \tau_j(B_k)$)

$$(17)\quad -\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & \cdots & 0 \\ \Delta\tau_1(B_2) & \cdots & \Delta\tau_1(B_g) \\ \vdots & \ddots & \vdots \\ \Delta\tau_{g-1}(B_2) & \cdots & \Delta\tau_{g-1}(B_g) \end{bmatrix} = \begin{bmatrix} -\iint_X \tau_1 \wedge \mu\tau_1 & \cdots & -\iint_X \tau_1 \wedge \mu\tau_{g-1} \\ \vdots & \ddots & \vdots \\ -\iint_X \tau_{g-1} \wedge \mu\tau_1 & \cdots & -\iint_X \tau_{g-1} \wedge \mu\tau_{g-1} \end{bmatrix}.$$

From this, we conclude that

$$(18)\quad \tau_j(\mu)(B_{k+1}) = \tau_j(B_{k+1}) + \iint_X \tau_j \wedge \mu\tau_k + O(\| \mu \|^2).$$

6.2. *A* 4–1 *cover of a Riemann surface*

When ρ is given by $\rho(A_j) = 1$ for all j, $\rho(B_1) = \rho(B_2) = -1$, $\rho(B_j) = 1$, for $3 \leqq j \leqq g$, the Prym differentials for ρ on X correspond to the following unbranched 4–1 cover of X. Let $\tilde{X}$ be the unbranched cover corresponding to the normal closure of B_1^2 and B_2^2 in $\pi_1(X)$. Then there are involutions on $\tilde{X}$ corresponding to the elements B_1 and B_2 of $\pi_1(X)$; these induce two involutions on $\mathrm{H}^{1,0}(\tilde{X})$. The projection of the combined $(-1, -1)$-eigenspace to X are the Prym differentials.

We can choose a basis for the Prym differentials $\tau_1, \ldots, \tau_{g-1}$ so the Prym matrix is

$$(19)\quad \begin{bmatrix} 1/2 & -1/2 & 0 & \cdots & 0 & -\tau_1(B_2) & \tau_1(B_2) & \tau_1(B_3) & \cdots & \tau_1(B_g) \\ 0 & 0 & 1 & \cdots & 0 & -\tau_2(B_2) & \tau_2(B_2) & \tau_2(B_3) & \cdots & \tau_2(B_g) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -\tau_{g-1}(B_2) & \tau_{g-1}(B_2) & \tau_{g-1}(B_3) & \cdots & \tau_{g-1}(B_g) \end{bmatrix}$$

The Prym bilinear relations for this Prym character reduce to $\tau_k(B_{j+1}) = \tau_j(A_{k+1})$ for all j and k, and the right $(g-1)\times(g-1)$ portion of this matrix has positive definite imaginary part. We can define the Prym variety as $\mathbf{C}^{g-1}$ divided by the lattice generated by the columns.

Now we substitute the entries of this matrix into the Prym variational formula. For this basis too, $T\tau_j = -\tau_j$, and so we obtain the matrix equation:

$$(20)\quad -\begin{bmatrix} 1/2 & -1/2 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \cdot \begin{bmatrix} -\Delta\tau_1(B_2) & \cdots & -\Delta\tau_1(B_g) \\ \Delta\tau_1(B_2) & \cdots & \Delta\tau_1(B_g) \\ \vdots & \ddots & \vdots \\ \Delta\tau_{g-1}(B_2) & \cdots & \Delta\tau_{g-1}(B_g) \end{bmatrix} = \begin{bmatrix} -\iint_X \tau_1 \wedge \mu\tau_1 & \cdots & -\iint_X \tau_1 \wedge \mu\tau_{g-1} \\ \vdots & \ddots & \vdots \\ -\iint_X \tau_{g-1} \wedge \mu\tau_1 & \cdots & -\iint_X \tau_{g-1} \wedge \mu\tau_{g-1} \end{bmatrix}.$$

The equation for each component is then

$$(21)\quad \tau_j(\mu)(B_{k+1}) = \tau_j(B_{k+1}) + \iint_X \tau_j \wedge \mu\tau_k + O(\|\mu\|^2).$$

Again, this is exactly the same form as that of the ordinary abelian case.

7. Some Prym differentials for surfaces of genus 2

Let us consider certain nonclassical Prym differentials for a surface X, differentials for a character ρ satisfying not $\rho^2 \equiv 1$, but $\rho^3 \equiv 1$. In this case, ρ^{-1} equals ρ^2, not ρ. This implies various connections among the Prym differentials and the Prym functions for ρ and ρ^{-1}. For example, the quotient of a Prym differential for ρ^{-1} and one for ρ is a Prym function for ρ.

Suppose now that the surface X is hyperelliptic, with genus g. Then X has an involution with $2g+2$ fixed points, called j, the hyperelliptic involution. The map j induces the map $j_*: H_1(X) \to H_1(X)$; this is simply multiplication by -1. Thus it induces the map $\rho \mapsto \rho^{-1}$ on the group of Prym characters for X, and thus j maps the Prym differentials for ρ^{-1} into those for ρ and vice versa.

For example, let X be the hyperelliptic surface of genus 2 given by $y^2 = \Pi_{i=1}^{6}(x-\lambda_i) = f(x)$. The involution j is the map $(x,y) \mapsto (x,-y)$. We label 8 special points on this surface as follows: we call the Weierstrass point $(\lambda_i, 0)$ Q_i, and we call the two points at infinity P_1 and P_2. Here, P_1 is the limit of (x,y) as $x \to \infty$ and $x^{-3}y \to 1$, while for P_2, $x^{-3}y \to -1$. We note that $jP_1 = P_2$ and $jQ_i = Q_i$ for all i.

Now we consider some important divisors on X.

$$(x)_\infty = P_1 + P_2,$$

$$(y) = Q_1 + Q_2 + Q_3 + Q_4 + Q_5 + Q_6 - 3P_1 - 3P_2,$$

$$(dx) = Q_1 + Q_2 + Q_3 + Q_4 + Q_5 + Q_6 - 2P_1 - 2P_2,$$

$$(dx/y) = P_1 + P_2.$$

When will an expression in x, y, and dx be a holomorphic Prym differential for some character ρ with $\rho^3 \equiv 1$? If τ is such a Prym differential, then τ^3 will be an ordinary cubic differential. We know that the space of holomorphic cubic differentials on X have a basis of

$$\{(dx/y)^3, x(dx/y)^3, x^2(dx/y)^3, x^3(dx/y)^3, y(dx/y)^3\}.$$

Therefore, up to multiplication of τ by a cube root of unity,

$$\tau^3 = (a_0 + a_1x + a_2x^2 + a_3x^3 + by)\left(\frac{dx}{y}\right)^3 \tag{22a}$$

and

$$\tau = \sqrt[3]{a(x)+by}\left(\frac{dx}{y}\right). \tag{22b}$$

Here, $a(x)$ is the cubic $a_0 + a_1x + a_2x^2 + a_3x^3$, and b is a complex constant. We now normalize this τ so that $b = 1$.

As τ is a Prym differential for ρ, $j\tau$ is a Prym differential for τ^{-1}. Therefore, $\tau \cdot j\tau$ is an ordinary holomorphic quadratic differential:

(23a) $$j\tau = \sqrt[3]{a(x) - y}\left(\frac{-dx}{y}\right),$$

(23b) $$\tau \cdot j\tau = -\sqrt[3]{a(x)^2 - y^2}\left(\frac{dx}{y}\right)^2,$$

(23c) $$= -\sqrt[3]{a(x)^2 - f(x)}\left(\frac{dx}{y}\right)^2.$$

However, the holomorphic quadratic differentials for X have the basis $\{(dx/y)^2, x(dx/y)^2, x^2(dx/y)^2\}$. Therefore,

(24) $$\sqrt[3]{a(x)^2 - f(x)} = R(x),$$

where $R(x)$ is at most quadratic. This implies that

(25) $$a(x)^2 - f(x) = R(x)^3.$$

Any solution of this equation (a, R) where a is at most cubic and R at most quadratic therefore induces a Prym differential $\sqrt[3]{a(x) + y}(dx/y)$. Since the Jacobian $J(X)$ has exactly 80 points of order 3, there are 80 different Prym differentials with $b = 1$. As R has an ambiguity of a cube root of 1, we have the following results.

LEMMA 4. *Let τ be a Prym differential for ρ, where $\rho^3 \equiv 1$ but $\rho \not\equiv 1$. Let $(\tau) = ST$ be the divisor of τ. Then neither S nor T may be a Weierstrass point.*

PROOF. Suppose $S = Q_i$, while $T \neq Q_i$. Then

$$\begin{aligned}(\tau) &= ST \\ &= Q_i T, \\ \left(\tau \Big/ \frac{dx}{y}\right) &= \frac{Q_i T}{P_1 P_2} \\ &\sim \frac{Q_i T}{Q_i^2} \\ &= \frac{T}{Q_i}.\end{aligned}$$

Therefore, (T^3/Q_i^3) is the divisor of a meromorphic function, and so 3 is a non-gap value for Q_i. Therefore, Q_i is not a hyperelliptic Weierstrass point; it does not have the correct gap sequence. Finally, if $S = T = Q_i$, then

$$\left(\tau \Big/ \frac{dx}{y}\right) \sim 1,$$

and τ is an abelian differential, violating the hypothesis $\rho \not\equiv 1$. ■

THEOREM 3. *Let X be the Riemann surface with defining equation $y^2 = f(x)$, where f is a sextic with complex coefficients, with distinct roots, and with leading coefficient* 1. *Then each nontrivial Prym character $\rho : \pi_1(X) \to \mathbf{C}^*$ with all values cube roots of unity is associated to a Prym differential with the form*

$$\tau = \sqrt[3]{a(x) + y}\left(\frac{dx}{y}\right)$$

where $a(x)$ is a polynomial of degree at most 3. *Furthermore,*

$$a(x)^2 - f(x) = R(x)^3$$

where R is a polynomial of degree at most 2. *If R is quadratic, let x_1 and x_2 be its roots. Then $(\tau) = (x_1, -a(x_1)) + (x_2, -a(x_2))$. If R is linear, with root x_0, then $(\tau) = (x_0, -a(x_0)) + P_i$, where P_i is P_1 or P_2 depending upon whether the lead term of $a(x)$ is $-x^3$ or $+x^3$. If R is constant, then $(\tau) = 2P_i$, with P_i defined as above.*

COROLLARY 1. *The equation $a(x)^2 - f(x) = R(x)^3$ has exactly* 240 *solutions for a and R.*

PROOF. Suppose R is a quadratic, with roots x_1 and x_2. Then $R(x) = c(x - x_1)(x - x_2)$, and this has zero divisor $(x_1, y_1) + (x_1, -y_1) + (x_2, y_2) + (x_2, -y_2)$, where $\pm y_i$ are the two square roots of $f(x_i)$ for each i. Furthermore, R has double poles at both P_1 and P_2. Therefore:

$$(R) = (x_1, y_1) + (x_1, -y_1) + (x_2, y_2) + (x_2, -y_2) - 2P_1 - 2P_2,$$

$$\left(R\left(\frac{dx}{y}\right)^2\right) = (x_1, y_1) + (x_1, -y_1) + (x_2, y_2) + (x_2, -y_2)$$

$$= (\tau \cdot j\tau).$$

As τ and $j \cdot \tau$ have degree 2, and $(j \cdot \tau) = j(\tau)$,

$$(\tau) = (x_1, \pm y_1) + (x_2, \pm y_2),$$

for some correct choices of sign. However, τ is zero when $y(x) = -a(x)$, and this case follows. If R is linear with root x, then $(x, -a(x))$ is certainly in the zero divisor of τ. Also, since R^3 has degree 3, $a(x)$ has leading coefficient ± 1. Now, divide the equation $a(x)^2 - f(x) = R(x)^3$ by x^6:

$$\frac{a(x)^2}{x^6} - \frac{f(x)}{x^6} = \frac{R(x)^3}{x^6},$$

$$\left(\frac{a(x)}{x^3}\right)^2 - \frac{f(x)}{x^6} = \left(\frac{R(x)}{x^2}\right)^3,$$

$$\frac{a(x)^2}{x^6} - (1 + O(x^{-1})) = O(x^{-3}),$$

$$\left(\frac{a(x)}{x^3} - 1\right)\left(\frac{a(x)}{x^3} + 1\right) + O(x^{-1}) = O(x^{-3}),$$

$$\left(\frac{a(x)}{x^3} - 1\right)\left(\frac{a(x)}{x^3} + 1\right) = O(x^{-1}).$$

Thus, if the leading coefficient is $+1$, then a has leading term x^3, and P_2 is the infinite root, and so on. Finally, if R is constant, τ must have 2 infinite roots. Since they cannot be both P_1 and P_2, in which case τ would be abelian, the roots must be P_1^2 or P_2^2. ■

Now, how do we associate the Prym differentials for ρ_1 and ρ_2 with the Prym differential for $\rho_1\rho_2$? We assume that $\rho_1 \neq \rho_2$ and $\rho_1 \neq \rho_2^{-1}$. Let

$$\tau_1 = \sqrt[3]{a_1(x) + y}\,\frac{dx}{y},$$

$$\tau_2 = \sqrt[3]{a_2(x) + y}\,\frac{dx}{y},$$

$$R_1 = \sqrt[3]{a_1(x)^2 - f(x)},$$

$$R_2 = \sqrt[3]{a_2(x)^2 - f(x)},$$

$$(\tau_1) = S_1 T_1,$$

$$(\tau_2) = S_2 T_2.$$

By the Riemann–Roch theorem,

$$\begin{aligned} r(3K - S_1 - T_1 - S_2 - T_2) &= \deg(3K - S_1 - T_1 - S_2 - T_2) - g + 1 \\ &\quad + r(S_1 + T_1 + S_2 + T_2 - 2K) \\ &= 2 - 2 + 1 + 0 \\ &= 1, \end{aligned}$$

since $S_1T_1S_2T_2$ is not the divisor of any quadratic differential. Therefore the space of all cubic differentials having $\tau_1\tau_2$ as a factor is one-dimensional. Let ω_3 be a generator of the space.

Since $\omega_3/\tau_1\tau_2$ is holomorphic, it is a Prym differential with character $(\rho_1\rho_2)^{-1}$. In fact, the cubic differential ω_3 is determined by:

Let

$$\tau_3 = \frac{\omega_3}{\tau_1\tau_2}.$$

Suppose

$$\tau_3 = \sqrt[3]{a_3(x) + y}.$$

Then

$$\tau_3 = \sqrt[3]{-a_3(x) + y}$$

is the differential for $\rho_1\rho_2$. We then write:

$$\omega_3 = c(d(x) + y)\left(\frac{dx}{y}\right)^3,$$

$$\tau_1\tau_2\tau_3 = \omega_3,$$

$$\sqrt[3]{a_1(x) + y}\,\sqrt[3]{a_2(x) + y}\,\sqrt[3]{a_3(x) + y}\left(\frac{dx}{y}\right)^3 = c(d(x) + y)\left(\frac{dx}{y}\right)^3,$$

$$\sqrt[3]{a_1(z) + y}\,\sqrt[3]{a_2(x) + y}\,\sqrt[3]{a_3(x) + y} = c(d(x) + y),$$

$$(a_1(x) + y)(a_2(x) + y)(a_3(x) + y) = c^3(d(x) + y)^3,$$

where $d(x)$ is a cubic in x. Now, $(d(x) + y)$ is zero at S_1, T_1, S_2, T_2. Since $(a_1(x) + y)$ is zero at S_1 and T_1, so is $d(x) - a_1(x)$. However, $R_1(x)$ is zero at S_1 and T_1. Therefore, $R_1(x) \mid d(x) - a_1(x)$. Similarly, $R_2(x) \mid d(x) - a_2(x)$. Now, $d(x)$ is unique; therefore it may be obtained by the Chinese remainder theorem, and $R_1(x)$ and $R_2(x)$ are relatively prime. (Actually, this follows from the Riemann–Hurwitz theorem, because otherwise $(\tau_1/\tau_2)^3$ would be a meromorphic function of degree 3 with branching order 4, implying that $g = 0$.) Therefore, we can compute $d(x)$ effectively.

Now, $(d(x) + y)$ has two more zeros besides S_1, T_1, S_2, T_1. We can find them without solving algebraically. Consider the equation

$$d(x)^2 - y^2 = d(x)^2 - f(x) = 0.$$

This is zero at the x-components of S_1, T_1, S_2, T_2, and the zeros of τ_3. Use the equations for the sum and product of the roots to find the x-components of the zeros of τ_3. Substitute into the equation $d(x) + y = 0$ to find y, and therefore τ_3. We may then change the sign of y to find τ_3. The case where P_1 and P_2 are roots of the differentials τ_1, τ_2, or their product corresponds to the case where $y + d(x)$ has fewer than six finite solutions; then $f(x) - d(x)^2$ has degree less than six, and thus there are additional conditions on d; we adjoin these conditions to the above, and the solution is then unique.

EXAMPLE 1. Let X be the Riemann surface $y^2 = x^6 - 1$. We write $f(x) = x^6 - 1$. Then we have $(x^3)^2 - f(x) = 1^3$ and $(i)^2 - f(x) = (-x^2)^3$. Therefore,

$$\tau_1 = \sqrt[3]{x^3 + y}\,\frac{dx}{y} \quad \text{and} \quad \tau_2 = \sqrt[3]{i + y}\,\frac{dx}{y}$$

are Prym differentials for two characters of the appropriate type on X. Now, $(\tau_1) = 2P_2$ and $(\tau_2) = 2(0, -i)$. To solve the problem

$$\tau_1\tau_2\tau_3 = \omega_3,$$

we see that $\omega_3 = (y + d(x))(dx/y)$ has at most $6 - 2 = 4$ finite zeros, and therefore

(1) $d(x)^2 - f(x)$ is at most a quartic,

(2) $d(x) - a_1(x)$ has a double zero at P_2,

(3) $d(x) - a_2(x)$ has a double zero at $(0, -i)$.

By the first condition, $d(x)$ must begin with the term x^3. By the third condition, $d(x) - i = x^3 + d_2x^2$ and so $f(x) = x^3 + d_2x^2 + i$. Since $d(x)^2 - f(x)$ is a quartic, $d_2 = 0$, and so $d(x) = x^3 + i$. Therefore $\omega_3 = (y + x^3 + i)(dx/y)^3$. Since $d(x)^2 - f(x) = 2ix^3$, ω_3 has three infinite roots and three roots when $x = 0$. Obviously, ω_3 has no root at $(0, i)$, and at an infinite root, $y/(-x^3)$ must tend to one. Thus, $(\omega_3) = 3P_2 + 3(0, -1)$, $(\tau_3) = P_2 + (0, -1)$, and so $(\tau_3) = P_1 + (0, i)$. Finally,

$$\tau_3 = \sqrt[3]{y - x^3 - 1}\left(\frac{dx}{y}\right).$$

We can give the following interpretation to this. By the Jacobi inversion theorem, there is a map

$$X \times X/\sim \to J(X),$$

with $(p, q) \sim (q, p)$ given by

$$(p, q) \mapsto \left(\int_{2p_0}^{p+q} \omega_1, \int_{2p_0}^{p+q} \omega_2\right),$$

where p_0 is an arbitrarily chosen base point of X. This map is bijective except over the point of $J(X)$ corresponding to the divisor class of abelian differentials. We can rid ourselves of this ambiguity as follows: Suppose ρ_1 corresponds to $p_1 + q_1$ and ρ_2 corresponds to $p_2 + q_2$ and $\rho_1 \neq \rho_2^{-1}$. Then, the product $\rho_1\rho_2$ corresponds to the divisor class $p_1 + q_1 + p_2 + q_2 - K$, and this is unique. When $\rho_1 = \rho_2^{-1}$, there is an ambiguity of a $\mathbf{CP}^1$. If we perform a blow-down on this locus, we get an abelian variety; this is precisely the Jacobian of X with base point one-half the abelian differential locus.

An alternative interpretation is the following; let X_3 be the tricanonical variety of X; this is a subset of $\mathbf{CP}^4$. The space of cubic differentials is 5-dimensional; therefore we have the map

$$X \hookrightarrow X_3 \subset \mathbf{CP}^4.$$

We have a standard basis for the cubic differentials; this exhibits X_3 as a 2–1 branched cover of the twisted cubic $x_1^3 = x_0x_1x_2 = x_0^2x_3$. Therefore, X_3 has degree six, and a general hyperplane strikes X_3 in 6 points.

We now have an "almost-group law", defined as follows. We can choose a particular canonical divisor $k_1 + k_2$. If τ_1 and τ_2 correspond to the divisors $S_1 + T_1$ and $S_2 + T_2$ respectively and $S_1 + T_1 + S_2 + T_2$ is not the divisor of a quadratic differential, then there is a unique hyperplane passing through S_1, T_1, S_2, and T_2. Assume that it passes through S_3 and T_3; then there is a unique hyperplane passing through S_3, T_3, k_1, and k_2. This passes through two more points, say S_4 and T_4, and then we have the following:

(26) $$S_1 + T_1 + S_2 + T_2 = S_4 + T_4.$$

8. The first variation of the Prym periods of a hyperelliptic surface

Let X be a hyperelliptic surface of genus $g \geqq 3$. Its tricanonical variety, X_3 is a curve of degree $6g-6$ in $\mathbf{CP}^{5g-6}$. The hyperelliptic involution of X induces an involution J_3 of $\mathbf{CP}^{5g-6}$; if the surface X has equation $w^2 = p(x)$ for p of degree $2g+2$, then the space of cubic differentials have as basis

(27) $$\frac{(dz)^3}{w^3}, \ldots, \frac{z^{3g-3}(dz)^3}{w^3}, \frac{(dz)^3}{w^2}, \ldots, \frac{z^{2g-4}(dz)^3}{w^3}.$$

The involution multiplies the first $3g-2$ terms by -1 and leaves the remaining $2g-3$ terms invariant.

Now, let ρ_1 and ρ_2 be Prym characters such that $\rho_1\rho_2 \neq 1$. The Prym differentials corresponding to each of ρ_1 and ρ_2 form vector spaces of dimension $g-1$; we can choose an arbitrary nonzero τ_1 and τ_2 in each. Then the divisor of $\tau_1\tau_2$ has degree $4g-4$ but is not bicanonical; therefore, by the Riemann–Roch theorem, there is a web of hyperplanes in $\mathbf{CP}^{5g-6}$ of dimension $g-2$ containing the divisor $(\tau_1\tau_2)$. If H_1 and H_2 are two hyperplanes containing $(\tau_1\tau_2)$, then

$$(H_1) \equiv (H_2),$$

$$(H_1) - (\tau_1\tau_2) \equiv (H_2) - (\tau_1\tau_2).$$

These form a linear series of type g_{2g-2}^{g-2}; in other words, these divisors correspond to a non-canonical class of degree $2g-2$. Therefore, they form the Prym differentials for some character ρ_3; certainly $\rho_3 = (\rho_1\rho_2)^{-1}$.

Now, make an arbitrary choice of a Prym differential with character ρ_3, and call it τ_3. $J\tau_3$ has character $\rho_1\rho_2$; this implies that $\tau_3 \cdot J\tau_3$ is an ordinary quadratic differential. Therefore, there is a meromorphic abelian differential ω such that $\tau_1 \cdot \tau_2 \cdot \tau_3 = \tau_3 \cdot J\tau_3 \cdot \omega$. Let $\tau_3 = J\tau_3$; then we have

(28) $$\tau_1\tau_2 = \tau_3\omega.$$

We then have an "almost group law" on the Prym differentials, or on the g_{2g-2}^{g-2}.

Now, consider a Prym differential τ with general character ρ. We have seen that $\tau \cdot J\tau$ is an ordinary quadratic differential. In fact, since $\tau \cdot J\tau$ is invariant under J,

$$\tau \cdot J\tau = a_0 + a_1 z + \cdots + a_{2g-2} z^{2g-2} \left(\frac{dz}{w}\right)^2 .$$

Then, we have that both

$$\tau \Big/ \left(\frac{dz}{w}\right) \quad \text{and} \quad J\tau \Big/ \left(\frac{dz}{w}\right)$$

are multiplicative meromorphic functions, and their product is the polynomial $a(z) = a_0 + a_1 z + \cdots + a_{2g-2} z^{2g-2}$.

Let $\lambda_1, \ldots, \lambda_{2g-2}$ be the roots of the polynomial $a(z)$. Since J sends $\tau/(dz/w)$ into the negative of $J\tau/(dz/w)$, and vice versa, if $(\lambda_i, w(\lambda_i))$ is a root of $\tau/(dz/w)$, then $(\lambda_i, -w(\lambda_i))$ is a root of $J\tau/(dz/w)$.

In principle, one can then determine the Prym differentials for any ρ, for if ρ has all of its values roots of unity, τ is a root of a higher-order ordinary differential, and it can be described locally by Puiseaux series. If not, one can still analyze it by the following procedure: Embed the surface X in its Jacobian $J(X)$; then, lift the Jacobian to $\mathbf{C}^g$. The surface X lifts to its homology cover X_∞, and the differentials lift to Fourier series in the coordiantes of $\mathbf{C}^g$ satisfying many conditions on their coefficients.

Even if we cannot calculate the Prym differentials of X by force, we can still determine the first variation of their periods. We must find the map $T: \Lambda^{1,0}(X, \rho) \to \Lambda^{1,0}(X, \rho^{-1})$ explicitly. We know that T takes holomorphic Prym differentials for ρ to holomorphic Prym differentials for ρ^{-1}. Now, the hyperelliptic involution J induces a bijection from $\mathrm{H}^{1,0}(X, \rho)$ to $\mathrm{H}^{1,0}(JX, \rho \circ J)$ given by $J^*\tau(z) = \tau(Jz)$. Since J is a holomorphic map and since

$$\begin{aligned} J^*\tau(Sz) &= \tau(J(S)J(z)) \\ &= \rho(J(S))\tau \circ J(z) \\ &= \rho(S^{-1})\tau \circ J(z) \\ &= \rho(S)^{-1}, \end{aligned}$$

J^* maps $\mathrm{H}^{1,0}(X, \rho)$ to $\mathrm{H}^{1,0}(X, \rho^{-1})$. As both J^* and T are linear, and both are bijections, there is a map M from $\mathrm{H}^{1,0}(X, \rho^{-1})$ to itself such that $M \circ J^* = T$. As J^* and T are bijective, M is too; therefore, the Prym differentials $\{T\tau_1, \ldots, Tt_{g-1}\}$ are linear combinations of the components of the other basis $\{J^*\tau_1, \ldots, J^*\tau_{g-1}\}$.

In [4], the effect of the hyperelliptic involution J on the fundamental group $\pi_1(X)$ is shown to be:

$$A_k \overset{J}{\mapsto} \prod_{l=k}^{g} [B_g \cdots B_l A_l, B_l] B_l \cdot A_k^{-1} \cdot \prod_{l=k}^{q} B_l^{-1}, \tag{29a}$$

$$B_k \overset{J}{\mapsto} B_g \cdots B_k A_k B_k^{-1} A_k^{-1} B_k^{-1} \cdots B_g^{-1}. \tag{29b}$$

In fact, these can be written as:

$$A_k \overset{J}{\mapsto} B_g \cdots B_k C_k \cdots C_g A_k^{-1} B_k^{-1} \cdots B_g^{-1}, \tag{30a}$$

$$B_k \overset{J}{\mapsto} B_g \cdots B_{k+1} C_k^{-1} B_k^{-1} \cdots B_g^{-1}. \tag{30b}$$

We can use our results on the Prym periods of products, conjugates, and commutators to show that if $\tau \in \mathrm{H}^{1,0}(X, \rho)$, then J acts on the Prym periods by

$$\begin{aligned} J^*\tau(A_k) &= \tau(JA_k) \\ &= \rho(B_g \cdots B_k)[\tau(C_k) + \cdots + \tau(C_g) + \tau(A_k^{-1})] \\ &\quad + (1 - \rho(A_k)^{-1})\tau(B_g \cdots B_k), \end{aligned} \tag{31a}$$

$$\begin{aligned} J^*\tau(B_k) &= \tau(JB_k) \\ &= \rho(B_g \cdots B_{k+1})[(1 - \rho(A_k) - \rho(B_k)^{-1})\tau(B_k) - (1 - \rho(B_k))\tau(A_k)] \\ &\quad + (1 - \rho(B_k)^{-1})\tau(B_g \cdots B_{k+1}). \end{aligned} \tag{31b}$$

The equations these formulas lead to seem intractable; however, we can restrict ourselves to the special case we have considered above, where $\rho(A_k) = 1$ for all k. In this case, the equations above reduce to:

$$\begin{aligned} J^*\tau(A_k) &= \tau(JA_k) \\ &= \rho(B_g \cdots B_k)[(1 - \rho(B_k))\tau(A_k) + \cdots + (1 - \rho(B_g))\tau(A_g) - \tau(A_k)] \\ &= \rho(B_g \cdots B_k)[-\rho(B_k)\tau(A_k) + (1 - \rho(B_{k+1}))\tau(A_{k+1}) \\ &\quad + \cdots + (1 - \rho(B_g))\tau(A_g)], \end{aligned} \tag{32a}$$

$$J^*\tau(B_k) = \tau(JB_k)$$

$$= \rho(B_g \cdots B_k)[-\rho(B_k)^{-1}\tau(B_k) - (1-\rho(B_k))\tau(A_k)] \tag{32b}$$

$$+ (1-\rho(B_k)^{-1})\tau(B_g \cdots B_{k+1}).$$

It is then true, but tedious to verify, that the following holds in both the general and the special cases:

$$\sum_{k=1}^{g} [(1-\rho(JB_k))\tau(JA_k) - (1-\rho(JA_k))\tau(JB_k)]$$

$$= \rho(B_g \cdots B_1) \sum_{l=1}^{g} \tau(C_l) \tag{33}$$

$$= 0.$$

Now, we recall that $T\tau$ is the differential in $\mathrm{H}^{1,0}(X, \rho^{-1})$ such that $T\tau(A_k) = -\overline{\tau(A_k)}$ for all k. Let $\sigma \in \mathrm{H}^{1,0}(X, \rho^{-1})$; then J affects the periods of σ the same way as it does the elements of $\mathrm{H}^{1,0}(X, \rho)$, except that ρ must be replaced by its inverse. Thus,

$$J^*\sigma(A_k) = \sigma(JA_k)$$

$$= \rho(B_g \cdots B_k)^{-1}[\sigma(C_k) + \cdots + \sigma(C_g) + \sigma(A_k^{-1})] \tag{34a}$$

$$+ (1-\rho(A_k))\sigma(B_0 \cdots B_k),$$

$$J^*\sigma(B_k) = \sigma(JB_k)$$

$$= \rho(B_g \cdots B_{k+1})^{-1}[(1-\rho(A_k)^{-1} - \rho(B_k))\sigma(B_k) - (1-\rho(B_k)^{-1})\sigma(A_k)] \tag{34b}$$

$$+ (1-\rho(B_k))\sigma(B_g \cdots B_{k+1}).$$

Again, in the special case where $\rho(A_k) = 1$ for all k,

$$J^*\sigma(A_k) = \sigma(JA_k)$$

$$= \rho(B_g \cdots B_k)^{-1}[(1-\rho(B_k)^{-1})\sigma(A_k)$$

$$+ \cdots + (1-\rho(B_g)^{-1})\sigma(A_g) - \sigma(A_k)] \tag{35a}$$

$$= \rho(B_g \cdots B_k)^{-1}[-\rho(B_k)^{-1}\sigma(A_k) + (1-\rho(B_{k+1})^{-1})\sigma(A_{k+1})$$

$$+ \cdots + (1-\rho(B_g)^{-1})\sigma(A_g)],$$

$$J^*\sigma(B_k) = \sigma(JB_k)$$

(35b)
$$= \rho(B_g \cdots B_{k+1})^{-1}[-\rho(B_k)\sigma(B_k) - (1-\rho(B_k)^{-1})\sigma(A_k)] + (1-\rho(B_k))\sigma(B_g \cdots B_{k+1}).$$

Now let $\sigma = T\tau$. Then $\sigma(A_k) = -\overline{\tau(A_k)}$, and so $J^*\sigma = J^*T\tau$ satisfies

$$J^*\sigma(A_k) = J^*T\tau(A_k)$$

(36a)
$$= \rho(B_g \cdots B_k)^{-1}[T\tau(C_k) + \cdots + T\tau(C_g) + \rho(A_k)^{-1}\overline{\tau(A_k)}] + (1-\rho(B_k))T\tau(B_g \cdots B_k),$$

$$J^*\sigma(B_k) = J^*T\tau(B_k)$$

(36b)
$$= \rho(B_g \cdots B_{k+1})^{-1}[(1-\rho(A_k)-\rho(B_k)^{-1})T\tau(B_k)] + (1-\rho(B_k)^{-1})\overline{\tau(A_k)} - T\tau(B_g \cdots B_{k+1}).$$

In the special case,

(37a)
$$J^*T\tau(A_k) = \rho(B_g \cdots B_k)^{-1}[-(1-\rho(B_k)^{-1})\overline{\tau(A_k)} - \cdots - (1-\rho(B_g)^{-1})\overline{\tau(A_g)} + \overline{\tau(A_k)}]$$
$$= \rho(B_g \cdots B_k)^{-1}[-\rho(B_k)^{-1}\overline{\tau(A_k)} - (1-\rho(B_{k+1})^{-1})\overline{\tau(A_{k+1})} - \cdots - (1-\rho(B_g)^{-1})\overline{\tau(A_g)}],$$

(37b)
$$J^*T\tau(B_k) = \rho(B_g \cdots B_{k+1})^{-1}[-\rho(B_k)T\tau(b_k) + (1-\rho(B_k)^{-1})\overline{\tau(A_k)}] - (1-\rho(B_k))\tau(B_g \cdots B_{k+1}).$$

In the special case, we can now solve the equation $T\tau = J\sigma$, for $\tau \in H^{1,0}(X, \rho)$. First, we take the action of J on the Prym periods of σ and express it as a matrix equation. By our results above, we have

$$(J\sigma(A_1), \ldots, J\sigma(A_g)) = (\sigma(A_1), \ldots, \sigma(A_g)) \cdot R,$$

where R is a matrix defined as follows:

(38)
$$R = \begin{bmatrix} -\rho(B_1)\rho(B_1, \ldots, B_g) & 0 & \cdots & 0 \\ (1-\rho(B_2))\rho(B_1, \ldots, B_g) & -\rho(B_2)\rho(B_2 \cdots B_g) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (1-\rho(B_g))\rho(B_1, \ldots, B_g) & (1-\rho(B_g))\rho(B_2 \cdots B_g) & \cdots & -\rho(B_g)^2 \end{bmatrix}.$$

Now, we know the equation

$$-\overline{\tau(A_k)} = J\sigma(A_k)$$

holds for all k. Therefore, we can apply this to our basis of Prym differentials (τ_k) as follows. Define σ_k as the solution of the above equation for each k. Then the differentials (σ_k) form a basis for the Prym differentials too. Thus, we may write

$$\sigma_k = a_{k1}\tau_1 + a_{k2}\tau_2 + \cdots + a_{kg-1}\tau_{g-1}.$$

Let M be the left half of the Prym matrix of the differentials σ_k and P be the left half of the period matrix of the differentials (τ_k). Then $M = AP$ and $-\bar{P} = MR$. From this, we conclude that $-\bar{P}R^{-1} = AP$. However, this is guaranteed to have a unique solution for A; it cannot be found explicitly because P is not square.

Since we now know what $T\tau$ is for any τ and for J, we can substitute into Theorem 2 above to find the first variation of the Prym differentials. If σ respects the hyperellipticity of X, then we can replace the integral with the action of J upon it, and thus we get an integral on the sphere.

Note added in proof. I have been informed that Corollary 1 above was proven using combinatorial methods by Hurwitz in [7]. Hurwitz proved this result by counting explicitly the number of ways a 3–1 cover of a surface can be built up, and how many different ways the homotopy group can act on $\mathbf{Z}_3$. Recently, this result has been revived as Tyrell's conjecture: Tyrell conjectures that the j-invariants of the 40 elliptic curves $y^2 = d(x)$ given from my construction are all distinct. I hope these methods will illuminate that conjecture.

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